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Kantowski–Sachs cosmological solution with torsion

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Abstract. We present an exact Kantowski–Sachs solution with torsion. The solution represents a magnetic model with stiff matter and is shown to be geodesically incomplete both to the past and the future.

1. Introduction

In recent years there has ben considerable interest in spatially homogeneous cosmological models. A space–time is said to be spatially homogeneous if it admits a group G_r of isometries acting transitively on space-like hypersurfaces; the isometry group may have 3, 4 or 6 parameters (cf Kramer *et al* 1980). If $r = 6$, the space–time is not only spatially homogeneous but also spatially isotropic, and belongs to the Friedmann–Robertson–Walker models. If $r = 4$, the space–time is locally rotationally symmetric (LRS) in the definition by Ellis (1967).

All Lie groups G_4 have three-parametric subgroups, whose orbits are either two or three dimensional. In the latter case, the space–time belongs to the homogeneous anisotropic Bianchi types I–IX (Ryan and Shepley 1975) and has been studied by many authors. In the former case, the orbits are necessarily of constant curvature. It has been proved by Kantowski (1966), that if this curvature is zero or negative the group G_4 admits a second three-parametric subgroup, whose orbits are three dimensional so we are back to the Bianchi models. However, no such subgroup is possible if the curvature is positive. This is the case I of Kantowski and Sachs (1966) and is thus known as the Kantowski–Sachs space–time.

The corresponding isometry group gives us the following metric for the Kantowski–Sachs universe

$$ds^2 = -dt^2 + S^2(t)\, dr^2 + R^2(t)(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

in spherically symmetric coordinates (t, r, θ, ϕ) and has been investigated by many authors in the general theory of relativity (GRT), following the studies by Kantowski and Sachs (see Collins (1977) for a detailed discussion of the global structure of such models in GRT, also for further references, and Vajk and Eltgroth (1970)). The extension of the Kantowski–Sachs models to the Einstein–Cartan theory of gravitation (ECT), in which the intrinsic spin of matter may avert the singularities that characterise GRT, was first given by Kuchowicz (1975, 1976a, b, c). However, the special solutions given by Kuchowicz are not of much physical value because of the *a priori* conditions

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on the metric functions R and S. It is the aim of this paper to construct an exact stiff matter solution of the ECT Kantowski–Sachs space–time in the presence of a magnetic field, which does not couple to torsion.

2. Field equations and solutions

The Kantowski–Sachs metric (1) is a special case of the metrics

$$ds^2 = -dt^2 + S^2(t)dr^2 + R^2(t)(d\theta^2 + f^2(\theta)\, d\phi^2) \tag{2}$$

where

$$f(\theta) := \begin{cases} \sinh\theta & \text{Bianchi type III} \\ \theta & \text{Bianchi type I} \\ \sin\theta & \text{Kantowski–Sachs} \end{cases} \tag{3}$$

(see MacCallum 1979a, b).

Defining

$$\delta := -\frac{f''(\theta)}{f(\theta)} = \begin{cases} -1 & \text{III} \\ 0 & \text{I} \\ 1 & \text{KS} \end{cases} \tag{4}$$

we obtain the ECT field equations for models with perfect fluid matter and magnetic field

$$\left(\frac{\dot R}{R}\right)^2 + 2\frac{\dot R}{R}\frac{\dot S}{S} + \frac{\delta}{S^2} = \varepsilon + \frac{f^2}{R^4} - s^2 \tag{5a}$$

$$\frac{\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2 + \frac{\dot R}{R}\frac{\dot S}{S} + \frac{\delta}{S^2} = \tfrac{1}{2}\varepsilon(2-\gamma) + \frac{f^2}{R^4} \tag{5b}$$

$$\frac{\ddot S}{S} + 2\frac{\dot R}{R}\frac{\dot S}{S} = \tfrac{1}{2}\varepsilon(2-\gamma) - \frac{f^2}{R^4} \qquad \dot{(\,)} = \frac{d}{dt} \tag{5c}$$

where the fluid is characterised by the equation of state

$$p = (\gamma - 1)\varepsilon \qquad 1 \leqslant \gamma \leqslant 2 \tag{6}$$

p is the pressure and ε the energy density of matter, f^2/R^4 are the components of the magnetic field, $\varepsilon = \varepsilon_0^2/(R^2S)^\gamma$ and $s = s_0/R^2S$ the spin density of matter (f^2, ε_0^2, s_0 are constants). We are thus considering aligned spin models of torsion.

We take the stiff ($\gamma = 2$) equation of matter. The possible relevance of the equation of state $p = \varepsilon$ as regards the matter content of the universe in its early stages has been discussed by a number of authors since it was first proposed by Zel'dovich (1961, 1970). We refer to the paper of Barrow (1978) for further discussions. Exact solutions for the magnetic ECT Bianchi type-I ($\delta = 0$) model have been given by Kuchowicz (1976b, d), Raychaudhuri (1975, 1979), Tsoubelis (1981) and Lorenz (1981). The magnetic Bianchi type-III ($\delta = -1$) case has been also considered by Tsoubelis (1981).

Introducing the new time variable τ by $\mathrm{d}t = R^2 S\, \mathrm{d}\tau$ the field equations to be solved are

$$(\ln RS)'' + (RS)^2 = 0 \tag{7a}$$

$$(\ln S)'' + (fS)^2 = 0 \tag{7b}$$

$$[(\ln RS)']^2 - [(\ln S)']^2 + (RS)^2 - (fS)^2 = \varepsilon_0^2 - s_0^2 \qquad (\;)' = \mathrm{d}/\mathrm{d}\tau. \tag{7c}$$

Equations (7*a*) and (7*b*) have as first integrals

$$[(\ln RS)']^2 + (RS)^2 = a \tag{8a}$$

$$[(\ln S)']^2 + (fS)^2 = b \tag{8b}$$

where a, b are constants of integration. From the constraint equation (7*c*) we obtain the condition

$$a = b + \varepsilon_0^2 - s_0^2. \tag{9}$$

Since R and S are real functions of τ, a and b must be positive. The corresponding solutions are

$$RS = k\, \mathrm{sech}(k\tau) \qquad a = k^2 > 0 \tag{10a}$$

$$S = \frac{l}{f}\, \mathrm{sech}(l\tau) \qquad b = l^2 > 0 \tag{10b}$$

where the integration constants were set equal to zero, as they only determine the point on the τ axis where the argument of the above functions vanishes.

3. Discussion

We now examine the behaviour of this new solution in some more detail. If $a = b$ we obtain an unrealistic model with one constant metric function. The case $a > b$ corresponds to $\varepsilon_0^2 > s_0^2$, so this solution reduces to a GRT model for vanishing torsion. However, the case $a < b$ represents a pure ECT Kantowski–Sachs model. It follows that these models are geodesically incomplete both to the past and the future and that the energy density of the fluid becomes infinite at both of these singularities. This is in contrast to the 'open' Bianchi type-III ECT solution, where it is possible to obtain a non-singular model (Tsoubelis 1981). We thus conclude that the behaviour of our solutions is very similar to that in which no torsion is present (see Collins 1977).

When no magnetic field is present, then it follows from equation (8*b*) that the expression (10*b*) is replaced by

$$S = S_0 \exp(\pm\tau) \qquad S_0 = \text{constant}. \tag{11}$$

Comparison of this solution with the previous one, where a magnetic field is present, shows the fact that such a field acts as a tension along its own direction. We should finally like to remark that the ratio ω/ε, where $\omega = f^2/R^4$ is the energy density of the magnetic field, approaches zero at both singularities in the magnetic ECT model.

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